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Discrete Mathematics 152 (1996) 287–293

DISCRETE MATHEMATICS

An identity involving 3-regular graphs

D.R. Woodall

Department of Mathematics, University of Nottingham, Nottingham, NG7 2RD, UK

Received 29 September 1993; revised 27 May 1994

Abstract

It is proved that, if M is a perfect matching in a 3-regular graph G, then the number of positive-minus-negative M-covers of G is equal to the number of positive-minus-negative M-partitions of G. Moreover, either there are no M-partitions of G, or every M-partition and every M-cover has the same sign.

This paper is devoted to the proof of a rather technical identity (Theorem 1), which is used in [2] to give a simplified method for calculating the discriminant of a bond graph. It is followed by a short theorem (Theorem 2) which gives more information about the quantities involved, and suggests the possible existence of a shorter proof of Theorem 1 — which, however, I have been unable to complete.

Let M be a perfect matching in a (finite) 3-regular graph $G = (V, E)$; we shall allow G to contain loops and parallel edges. The edges of M ($E - M$) are called *defining* (*non-defining*) in [2]; they are represented by thick (thin) lines in figures.

An *M-walk* in G (called a *coenergy loop* in [2], the term 'loop' being used there in its engineering sense of closed walk or feedback loop) is a closed walk W that traces alternately edges in M and $E - M$, and that traces each edge of M at most once in each direction and each edge of $E - M$ at most once in total. W is *proper* if it traces no edge twice, and *improper* otherwise. Suppose that each edge of G is given a *reference direction*. Since W (clearly) has even length, the number of edges that it traces in the reference direction has the same parity as the number that it traces in the opposite direction. The *sign* of W is (perversely) defined to be positive or negative according as this number is odd or even. (This definition of *sign* incorporates an extra minus sign that is implicit in the equations in [2].) The reverse $\bar{W}$ of W (that is, W traced in the reverse direction) is clearly an M-walk with the same sign as W.

For example, the M-walk in the graphs G_1 in Fig. 1, with their signs and proprieties, are as follows. (Here a_i denotes an edge traced in the reference direction—indicated by an arrow in Fig. 1—and $\bar{a}_i$ denotes the same edge traced in the

SSDI 0012-365X(94)00274-6

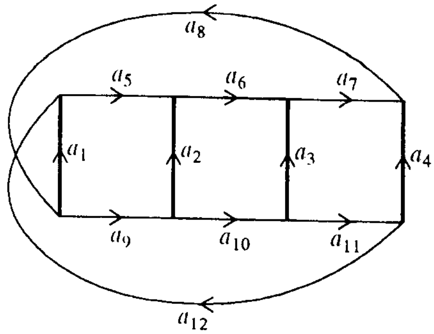

Fig. 1.

opposite direction.)

$-$	proper	$a_1a_5\bar{a}_2\bar{a}_9$	$a_9a_2\bar{a}_5\bar{a}_1$
$-$	proper	$a_2a_6\bar{a}_3\bar{a}_{10}$	$a_{10}a_3\bar{a}_6\bar{a}_2$
$-$	proper	$a_3a_7\bar{a}_4\bar{a}_{11}$	$a_{11}a_4\bar{a}_7\bar{a}_3$
$+$	proper	$a_4a_8a_1\bar{a}_{12}$	$a_{12}\bar{a}_1\bar{a}_8\bar{a}_4$

$$- \quad \text{improper} \begin{cases} a_1a_5\bar{a}_2a_{10}a_3a_7\bar{a}_4a_{12}\bar{a}_1a_9a_2a_6\bar{a}_3a_{11}a_4a_8 \\ \bar{a}_8\bar{a}_4\bar{a}_{11}a_3\bar{a}_6\bar{a}_2\bar{a}_9a_1\bar{a}_{12}a_4\bar{a}_7\bar{a}_3\bar{a}_{10}a_2\bar{a}_5\bar{a}_1 \end{cases}$$

An *M-cover* of G is a set of M-walks that, between them, trace every edge of M exactly once in each direction and every edge of $E - M$ exactly once in total. Its *sign* is the product of the signs of all the M-walks in it. The graph G_1 in Fig. 1 has two M-covers, both negative, since each consists of a single negative improper M-walk. Let $C(G, M)$ denote the number of positive M-covers of G minus the number of negative M-covers of G; thus $C(G_1, M) = 2$. Note that $C(G,M)$ is always even, since if $\mathscr{C}$ is an M-cover, then so is $\{\bar{W}\colon W \in \mathscr{C}\}$, and it has the same sign as $\mathscr{C}$.

A *proper M-track* in G (called a *proper causal loop* in [2]) is the set of edges (with no associated directions) of a proper M-walk W in G, and has the same sign as W; thus the two distinct proper M-walks W and $\bar{W}$ correspond to the same proper M-track. For example, in G_1, the two negative proper M-walks $a_1a_5\bar{a}_2\bar{a}_9$ and $a_9a_2\bar{a}_5\bar{a}_1$ both correspond to the same negative proper M-track $\{a_1, a_2, a_5, a_9\}$. An *M-partition* of G is an ordered pair (A, B), where each of A and B is a set of proper M-tracks that contain every edge of M exactly once, and the M-tracks of A and B together contain every edge of $E - M$ exactly once. The *sign* of this M-partition is the product of the signs of all the M-tracks in A and B. Thus G_1 has two M-partitions, one of which is

$$A = \{\{a_1, a_2, a_5, a_9\}, \{a_3, a_4, a_7, a_{11}\}\},$$

$$B = \{\{a_2, a_3, a_6, a_{10}\}, \{a_1, a_4, a_8, a_{12}\}\},$$

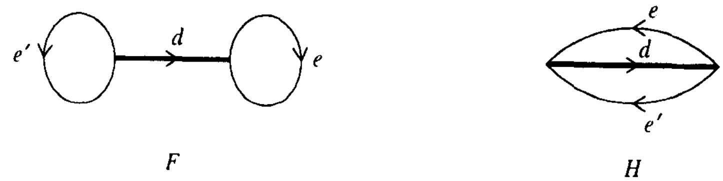

Fig. 2.

the other being obtained from this by interchanging A and B; both are negative. Let $P(G, M)$ denote the number of positive M-partitions of G minus the number of negative M-partitions of G; thus $P(G_1, M) = 2$. Note that $P(G, M)$ is always even, since if (A, B) is an M-partition, then so is (B, A), and it has the same sign as (A, B).

The main result of this note is the following identity.

Theorem 1. $C(G, M) = P(G, M)$.

Proof. Note first that if we change the reference direction of an edge $e \in E - M$, then we change the sign of every M-walk and M-track that uses e, and hence we merely change the sign of both $C(G, M)$ and $P(G, M)$, since each M-cover or M-partition contains exactly one M-walk or M-track, respectively, that uses e. Thus we may choose the reference directions of edges in $E - M$ to suit our convenience. Note also that it suffices to prove the result for connected graphs, since both $C(G, M)$ and $P(G, M)$ are multiplicative over components.

We shall prove the result by induction on $|M|$. If $|M| = 1$, then we may suppose, in view of the previous paragraph, that G is one of the graphs F and H in Fig. 2. We note that each M-cover of F consists of a single (improper) M-walk; there are two positive M-covers (namely $\{de\bar{d}e'\}$ and its reverse, $\{\bar{e}'d\bar{e}\bar{d}\}$), and two negative M-covers ($\{de\bar{d}\bar{e}'\}$ and its reverse, $\{e'd\bar{e}\bar{d}\}$), and so $C(F, M) = 0$. Also $P(F, M) = 0$, since there are no proper M-walks in F.

In contrast, every M-walk in H is proper, and negative. There are two M-covers, $\{de, \bar{d}\bar{e}'\}$ and $\{de', \bar{d}\bar{e}\}$, and so $C(H, M) = 2$ (since an M-cover containing two negative M-walks is positive). Also there are two M-partitions, $(\{\{d, e\}\}, \{\{d, e'\}\})$ and $(\{\{d, e'\}\}, \{\{d, e\}\})$, and so $P(H, M) = 2$. Thus the result holds for both graphs with $|M| = 1$.

Suppose now that $|M| \geqslant 2$. There are three cases to consider.

Case 1: G contains the configuration shown in Fig. 3 (after appropriate choices of reference directions of edges in $E - M$), where e' and e'' are distinct edges since we are supposing that G is connected and $|M| \geqslant 2$. Consider the graph G' obtained by removing d, e_0 and their incident vertices, and merging e' and e'' into a single edge e. Then G' is a 3-regular graph, and $M' = M \cap G'$ is a perfect matching of G' with $|M| - 1$ edges. Let $\mathscr{C}'$ be an *M'-cover* of G'. Exactly one of $e, \bar{e}$ is contained in an M'-walk in $\mathscr{C}'$. If e is, then we can replace it by either $e'\bar{d}e_0de''$ or $e'\bar{d}\bar{e}_0de''$ to give two

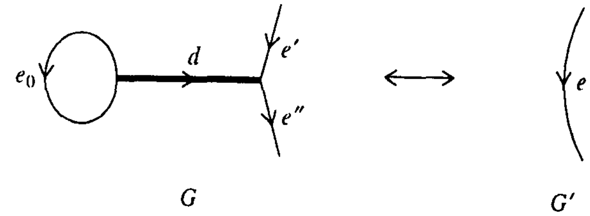

Fig. 3.

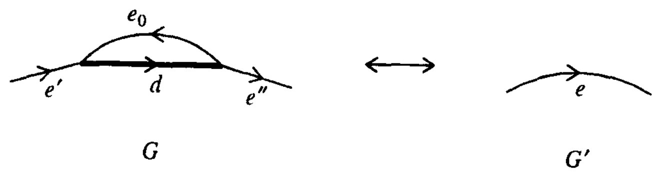

Fig. 4.

M-covers of G with opposite signs; similarly, if $\bar{e}$ is, then we can replace it by either $\bar{e}''\bar{d}\bar{e}_0 d\bar{e}'$ or $\bar{e}''\bar{d}e_0 d\bar{e}'$ to give, again, two M-covers of G with opposite signs. Since every M-cover of G is obtained from exactly one M'-cover of G' in this way, it follows that the number of positive M-covers of G is equal to the number of negative M-covers, that is, $C(G,M) = 0$. But $P(G,M) = 0$ also, since d is not contained in any proper M-walks and so there can be no M-partitions.

Case 2: G contains the configuration shown in Fig. 4, where again e' and e'' are distinct edges. As before, consider the graph G' obtained by removing d, e_0 and their incident vertices, and merging e' and e'' into a single edge e. Then G' is a 3-regular graph, and $M' = M \cap G'$ is a perfect matching of G' with $|M| - 1$ edges. This time, each M'-cover of G' gives rise to exactly one M-cover of G, with the opposite sign. For, let $\mathscr{C}'$ be an M'-cover of G'. Exactly one of $e, \bar{e}$ is contained in an M'-walk in $\mathscr{C}'$. If e is, then we replace it by $e'de''$ and add $\bar{d}\bar{e}_0$ as a new (negative) M-walk, and if $\bar{e}$ is, then we replace it by $\bar{e}''\bar{d}\bar{e}'$ and add de_0 as a new M-walk. Since every M-cover of G is obtained from exactly one M'-cover of G' in this way, it follows that $C(G,M) = -C(G',M')$. Similarly, every M-partition of G is obtained from exactly one M'-partition $\mathscr{P}'$ of G' by replacing e, in the M'-track of $\mathscr{P}'$ containing it, by e', d and e'', and adding the (negative) proper M-track $\{d, e_0\}$ to the opposite part of the partition. Thus $P(G,M) = -P(G',M')$. Since we may suppose inductively that $C(G',M') = P(G',M')$, we can deduce that $C(G,M) = P(G,M)$, as required.

Case 3: Neither case 1 nor case 2 arises. Then we can ensure, by appropriate choices of reference directions, that G contains the configuration shown in Fig. 5, where all edges shown are distinct. Consider the two graphs G' and G'' formed from G as shown in Fig. 5, and let $M' = M \cap G'$ and $M'' = M \cap G''$. Then G' and G'' are 3-regular graphs containing perfect matchings M' and M'', respectively, and $|M'| = |M''| = |M| - 1$.

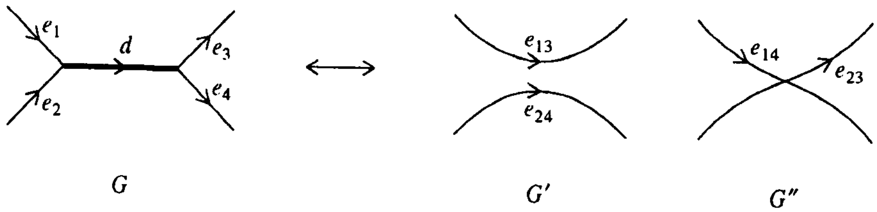

Fig. 5.

Every M-walk in G containing d corresponds in an obvious way to *either* an M'-walk in G' *or* an M''-walk in G'' (with the same sign), but not both. Conversely, every M'-walk in G' and every M''-walk in G'' corresponds in a natural way to a walk in G, with the same sign, that is either an M-walk or else fails to be an M-walk solely because it traces d twice in the same direction. Thus every M-cover of G corresponds to either an M'-cover of G' or an M''-cover of G'', with the same sign. And every M'-cover of G', and every M'' cover of G'', corresponds to *either* an M-cover of G, *or* to a set of walks in G that trace every edge of $E - M$ exactly once, every edge of $M - \{d\}$ exactly once in each direction, and d twice in the same direction. However, the sets of walks of this latter type pair off, the two sets in each pair having opposite sign. For, consider the sets of walks that trace d twice in its reference direction. (The argument is similar for the sets that trace d twice in the opposite direction.) Each of them must also trace two paths P, P' from the head of d to its tail, and for each set of walks that contains two distinct M-walks dP and dP' there is another set (with opposite sign) that is identical except that it contains instead the single walk $dPdP'$; and vice versa. Thus these unwanted sets of M-walks cancel out, and we have $C(G,M) = C(G',M') + C(G'',M'')$.

We have a similar problem with the M-partitions. Every M-partition of G corresponds in a natural way to either an M'-partition of G' or an M''-partition of G'', with the same sign. But an M'-partition of G' (say) may correspond to *either* an M-partition of G, *or* to a pair (A, B) where each of A and B is a set of M-tracks that contain every edge of $M - \{d\}$ exactly once, and the M-tracks of A and B together contain every edge of $E - M$ exactly once, but d is contained in two different M-tracks in A and none in B (say). This second, unwanted, possibility occurs when e_{13} and e_{24} are in M'-tracks in the same part of the M'-partition. However, these unwanted partitions cancel out, since, for each unwanted M'-partition of G', there is a corresponding unwanted M''-partition of G'' with the opposite sign. To see this, suppose that e_{13} and e_{24} are contained in two different M'-tracks S_1 and S_2 in the same part A of an M'-partition (A, B) in G'. Then $S_1 = P_1 \cup \{e_{13}\}$ and $S_2 = P_2 \cup \{e_{24}\}$, where P_1 and P_2 are the edge-sets of paths connecting the head of e_{13} to its tail and the head of e_{24} to its tail, respectively; thus $S = P_1 \cup P_2 \cup \{e_{14}, e_{23}\}$ is a single M''-track in G'', and $(A \cup \{S\} - \{S_1, S_2\}, B)$ is an M''-partition of G'' with the opposite sign from (A, B). Conversely, suppose that e_{13} and e_{24} are contained in the same M'-track S' in one part

A' of an M'-partition (A', B') in G'. Then $S' = P_1' \cup P_2' \cup \{e_{13}, e_{24}\}$, where P_1' and P_2' are the edge-sets of paths connecting the head of e_{24} to the tail of e_{13} and the head of e_{13} to the tail of e_{24}, respectively; thus $S_1' = P_1' \cup \{e_{14}\}$ and $S_2' = P_2' \cup \{e_{23}\}$ are two disjoint M''-tracks in G'', and $(A' \cup \{S_1', S_2'\} - \{S'\}, B')$ is an M''-partition of G'' with the opposite sign from (A', B'). It follows from this argument that all the unwanted partitions cancel out, and so $P(G, M) = P(G', M') + P(G'', M'')$.

Since we may suppose inductively that $C(G', M') = P(G', M')$ and $C(G'', M'') = P(G'', M'')$, it now follows that $C(G, M) = P(G, M)$, as required. This completes the proof of Theorem 1. $\square$

It is now easy to identify the common value of $C(G, M)$ and $P(G, M)$. Note that $E - M$ consists of a collection of disjoint circuits.

Theorem 2. *Suppose that $E - M$ consists of k disjoint circuits. If any of them has odd length, then G has 2^{k-1} positive and 2^{k-1} negative M-covers, and there are no M-partitions. If all of them have even length, then there are 2^k M-covers and 2^k M-partitions, all with the same sign.*

Proof. In tracing the M-walks in an M-cover of G, we trace all the edges in each circuit C of $E - M$ in the same direction around C. Conversely, if we choose a direction for each circuit C, then there is a corresponding M-cover of G. Since there are 2^k possible choices for the directions around the k circuits, it follows that there are 2^k different M-covers. We shall prove in the next paragraph that the number of M-walks in every M-cover of G has the same parity. Therefore, if any circuit C has odd length, then reversing the direction around C will change the sign of the M-cover; thus the M-covers pair off, there being 2^{k-1} positive and 2^{k-1} negative M-covers. But if all the circuits have even length, then all 2^k M-covers have the same sign.

To prove that the number of M-walks in every M-cover has the same parity, it suffices to consider an M-cover $\mathscr{C}$ and a circuit C, and to show that reversing the direction round C does not change the parity of the number of M-walks in $\mathscr{C}$. I am indebted to John Lamb [1] for the following proof of this. Label the vertices of C as $v_1, v_2, \ldots, v_k, v_1$ in order round C, so that in tracing the M-walks in $\mathscr{C}$ we trace each edge $v_i v_{i+1}$ of C from v_i towards v_{i+1}. Let $K = \{1, 2, \ldots, k\}$, let α denote the cyclic permutation $(1\,2 \ldots k)$ on K, and let β denote the permutation on K in which $\beta(i) = j$ if, in tracing the M-walk of $\mathscr{C}$ that contains the edge $v_{i-1} v_i$, the next vertex of C that we arrive at is v_j. Then the cycles in the permutation $\beta\alpha$ (α followed by β) correspond to the M-walks of $\mathscr{C}$ that intersect C, in the sense that edges $v_i v_{i+1}$ and $v_j v_{j+1}$ of C are contained in the same M-walk of $\mathscr{C}$ if and only if i and j are in the same cycle of the permutation $\beta\alpha$. The same is true for the permutation $\beta\alpha^{-1}$ if we reverse the direction around C. Thus we must prove that the permutations $\beta\alpha$ and $\beta\alpha^{-1}$ have the same number of cycles (mod 2). Clearly these permutations are both even or both odd. But, in any permutation on K, the number of cycles of odd length has the same parity as k, and so the total number of cycles has parity equal to k plus the number of cycles of even length, which is determined by k and the signature of the permutation. Thus $\beta\alpha$ and $\beta\alpha^{-1}$ have the same number of cycles (mod 2), which means that reversing the direction around C does not change the parity of the total number of M-walks that intersect C, and hence does not change the parity of the number of M-walks, since those that do not intersect C are unaffected by the reversal. This suffices to prove that the number of M-walks in every M-cover has the same parity, as required.

We now consider the M-partitions. Each M-partition (A, B) of G gives rise to an edge-2-colouring of $E - M$ by colouring all edges of $E - M$ in A amber and all edges of $E - M$ in B blue. Conversely, given an edge-2-colouring of $E - M$ there is a corresponding M-partition of G. If any circuit C in $E - M$ has odd length, then $E - M$ has no edge-2-colourings, and so G has no M-partitions. But every circuit of even length has two possible different edge-2-colourings, and so if all the circuits have even length, then G has 2^k different M-partitions. What is not obvious (to me) is that they all have the same sign; but this now follows from the previous paragraphs and Theorem 1. □

References

[1] J.D. Lamb, Theory of bond graphs, Ph.D. Thesis, Nottingham University, 1993.

[2] J.D. Lamb, D.R. Woodall and G.M. Asher, Bond graphs II: causality and singularity, Discrete Math., submitted.